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The eight-vertex model and Painlevé VI equation II: eigenvector results

Vladimir V Mangazeev and Vladimir V Bazhanov

Department of Theoretical Physics, Research School of Physical Sciences and Engineering, Australian National University, Canberra, ACT 0200, Australia

E-mail: Vladimir.Mangazeev@anu.edu.au and Vladimir.Bazhanov@anu.edu.au

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Abstract
We study a special anisotropic XYZ-model on a periodic chain of an odd length and conjecture exact expressions for certain components of the ground state eigenvectors. The results are written in terms of tau-functions associated with Picard's elliptic solutions of the Painlevé VI equation. Connections with other problems related to the eight-vertex model are briefly discussed.

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1. Introduction

This is a sequel to our paper [1] devoted to connections of the eight-vertex model of statistical mechanics [2] with the theory of Painlevé transcendents. Here we study a related case of the anisotropic XYZ-model on a periodic chain of an odd length, $N = 2n + 1$.

Let $\sigma_x^{(j)}$, $\sigma_y^{(j)}$ and $\sigma_z^{(j)}$, $j = 1, \ldots, N$, denote usual Pauli matrices acting at the jth site of the chain. Consider a particular XYZ-Hamiltonian

$$\mathbf{H}_{\mathsf{XYZ}} = -\frac{1}{2}\sum_{j=1}^{N}\left(J_x\sigma_x^{(j)}\sigma_x^{(j+1)} + J_y\sigma_y^{(j)}\sigma_y^{(j+1)} + J_z\sigma_z^{(j)}\sigma_z^{(j+1)}\right), \tag{1}$$

where the coefficients

$$J_x = \frac{2(1+\zeta)}{\zeta^2+3}, \qquad J_y = \frac{2(1-\zeta)}{\zeta^2+3}, \qquad J_z = \frac{\zeta^2-1}{\zeta^2+3} \tag{2}$$

are specific rational functions of a single parameter ζ, which satisfy the relation

$$J_xJ_y + J_yJ_z + J_zJ_x = 0. \tag{3}$$

Baxter [3] proved that for an infinitely large chain, $N \to \infty$, the ground state eigenvalue of (1) in this case has a very simple form:

$$\frac{E_0(\zeta)}{N} = -\frac{J_x + J_y + J_z}{2} = -\frac{1}{2}. \tag{4}$$

Later on it was conjectured [4] that this expression is exact for all finite odd values of N.

The Hamiltonian (1) commutes with the transfer matrix of the eight-vertex model (8V-model), where the Boltzmann weights a, b, c, d (we use standard notations of [2], see section 2 below for further details) are constrained as

$$(a^2+ab)(b^2+ab) = (c^2+ab)(d^2+ab), \tag{5}$$

and the variable ζ in (2) is given by

$$\zeta = \frac{cd}{ab}, \qquad \gamma = \frac{(a-b+c-d)(a-b-c+d)}{(a+b+c+d)(a+b-c-d)}. \tag{6}$$

An additional variable γ, introduced for later convenience, is connected to ζ by a simple self-reciprocal rational substitution:

$$\zeta = \frac{\gamma+3}{\gamma-1}, \qquad \gamma = \frac{\zeta+3}{\zeta-1}. \tag{7}$$

The spectrum of (1) possesses an S_3 symmetry group with respect to permutations of the constants J_x, J_y and J_z. Indeed, any such permutation can be compensated by a linear transformation which acts on the eigenvectors and does not affect the spectrum. For the parametrization (2) this group is generated by two substitutions of the variable ζ,

$$\begin{aligned} \mathsf{s}_{\mathsf{xy}}: \quad & \zeta \;\to -\zeta \quad && \Longrightarrow \quad J_x \leftrightarrow J_y,\; J_z \to J_z, \\ \mathsf{s}_{\mathsf{xz}}: \quad & \zeta \to \frac{\zeta+3}{\zeta-1} \quad && \Longrightarrow \quad J_x \leftrightarrow J_z,\; J_y \to J_y. \end{aligned} \tag{8}$$

The largest eigenvalue of the transfer matrix, corresponding to (4), also has a remarkably simple conjectured form [4]

$$\Lambda_0 = (a+b)^N, \qquad N = 2n+1, \tag{9}$$

which is expected to hold for finite chains.

In [5] we studied Baxter's famous TQ-equation for this simple eigenvalue (9) and found corresponding eigenvalues of the Q-operator. With an appropriate normalization they can be expressed through certain polynomials

$$\mathcal{P}_n(x,z) = \sum_{k=0}^{n} r_k^{(n)}(z) x^k, \qquad z = \gamma^{-2}, \qquad n = 0,1,2,\ldots, \tag{10}$$

of the variable x, defined by the following quadratic equation (see also (23)):

$$\left(\sqrt{x} - \frac{\gamma}{\sqrt{x}}\right)^2 = -\frac{16(a-b)^2 cd}{(c+d)^2(a+b+c+d)(a+b-c-d)}. \tag{11}$$

The coefficients $r_i^{(n)}(z)$, $i = 0,\ldots,n$, appearing in (10) are polynomials in the variable $z = \gamma^{-2}$ with positive integer coefficients. Detailed definitions of the polynomials (10) are presented in section 2. Here we want to illustrate their connection to the Painlevé VI equation. This connection manifests itself in some specific properties of the coefficients $r_i^{(n)}(z)$. In particular, let $s_n(z) \equiv r_n^{(n)}(z)$ be a coefficient in front of the leading power of x in (10). In [5] we conjectured the following recurrence relation:

$$\begin{aligned} &2z(z-1)(9z-1)^2\partial_z^2 \log s_n(z) + 2(3z-1)^2(9z-1)\partial_z \log s_n(z) \\ &\quad + 8(2n+1)^2\frac{s_{n+1}(z)s_{n-1}(z)}{s_n^2(z)} - [4(3n+1)(3n+2)+(9z-1)n(5n+3)] = 0, \end{aligned} \tag{12}$$

where $s_0(z) = s_1(z) \equiv 1$, which uniquely determines the polynomials $s_n(z)$, for all $n \in \mathbb{Z}$. Later on we proved [1] that equation (12) exactly coincides with the recurrence relation for the tau-functions associated with special elliptic solutions of the Painlevé VI equation. In this letter we extend these connections to study *ground state eigenvectors* of $\mathbf{H}_{\mathsf{XYZ}}$, corresponding to the eigenvalue (4).

For odd N all eigenvalues of (1) are double degenerate. Thus, there are two ground state eigenvectors

$$\mathbf{H}_{\mathsf{XYZ}}\Psi_\pm = E_0\Psi_\pm, \qquad \mathcal{S}\Psi_\pm = \pm\Psi_\pm, \qquad \mathcal{R}\Psi_\pm = \Psi_\mp, \tag{13}$$

where

$$\mathcal{S} = \sigma_z^{(1)} \otimes \sigma_z^{(2)} \otimes \cdots \otimes \sigma_z^{(N)}, \qquad \mathcal{R} = \sigma_x^{(1)} \otimes \sigma_x^{(2)} \otimes \cdots \otimes \sigma_x^{(N)}, \tag{14}$$

and

$$[\mathbf{H}_{\mathsf{XYZ}}, \mathcal{S}] = [\mathbf{H}_{\mathsf{XYZ}}, \mathcal{R}] = 0, \qquad \mathcal{RS} = (-1)^N \mathcal{SR}. \tag{15}$$

Due to the spin reversal symmetry, generated by the operator $\mathcal{R}$, it is enough to consider one of these vectors. For definiteness, consider the vector Ψ_-. Omitting the suffix '−', we denote its components as $\Psi_{i_1,i_2,\ldots,i_N}$, where $i_1, i_2, \ldots, i_N \in \{0, 1\}$ and assume an orthonormal basis $|i\rangle$, $i = 0, 1$, for each spin:

$$\sigma_z|0\rangle = +|0\rangle, \qquad \sigma_z|1\rangle = -|1\rangle. \tag{16}$$

The ground state eigenvectors are translationally invariant and possess a left–right reflection symmetry. Taking this into account, we will give only one non-vanishing representative from each symmetry class. Note that for non-vanishing components $\Psi_{i_1,i_2,\ldots,i_N}$ of Ψ_-, the number of 'down-spins' in the set $\{i_1, i_2, \ldots, i_N\}$ is odd, while for vanishing components it is even,

$$\Psi_{i_1,i_2,\ldots,i_N} \equiv 0, \qquad \text{if} \quad i_1 + i_2 + \cdots + i_N = 0 \pmod 2. \tag{17}$$

The fact that both the coefficients (2) and the eigenvalue (4) are *rational functions in* ζ *with integer coefficients* implies that with a suitable normalization one can make all components of the eigenvector Ψ_- to be *polynomials in* ζ *with integer coefficients* [6] (such that there are no polynomial factors common for all components). This choice is unique up to a numerical normalization. The latter is fixed by the requirements

$$\Psi_{\underbrace{0,\ldots,0}_{n+1}\underbrace{1,\ldots,1}_{n}}\Big|_{\zeta=0} = 1, \qquad \text{for odd } n; \qquad \Psi_{\underbrace{0,\ldots,0}_{n}\underbrace{1,\ldots,1}_{n+1}}\Big|_{\zeta=0} = 1, \qquad \text{for even } n. \tag{18}$$

Note that in the case $\zeta = 0$, the Hamiltonian (1) reduces to that of the XXZ-model with the parameter $\Delta = -1/2$. From this point of view, the normalization (18) is identical to that used in [7], where this particular XXZ-model was studied.

We have calculated all components of the eigenvectors directly from definition (13) for $N \leqslant 17$ (and some particular components for $N \leqslant 25$) and made several interesting observations which we formulate as conjectures valid for all $N = 2n + 1$. As an example we present here

Conjecture 1. *The norm of the eigenvector* Ψ_- *is given by*

$$|\Psi_-|^2 = \sum_{i_1,i_2,\ldots,i_N\in\{0,1\}} \Psi^2_{i_1,i_2,\ldots,i_N} = (4/3)^n \zeta^{n(n+1)} s_n(\zeta^{-2}) s_{-n-1}(\zeta^{-2}), \tag{19}$$

where $s_n(\zeta^{-2})$, $n \in \mathbb{Z}$, *are defined by the recurrence relation (12) with* $z = \zeta^{-2}$ *and* $s_0(z) = s_1(z) \equiv 1$.

Other conjectures on the properties of eigenvectors require additional notations; they are presented in section 3. Basic definitions for the 8V-model, a brief review of some of our previous results [5] and some new results on the eigenvalues of Baxter's Q-operators are given in section 2. In section 4, we discuss some unresolved questions and connections of our results to other problems related with the eight-vertex model, in particular, to the eight-vertex solid-on-solid model [8] with the domain wall boundary condition [9] and the three-coloring problem [10].

2. The eight-vertex model and TQ-equation

2.1. Basic definitions and notations

We consider the eight-vertex model on the N-column square lattice with the periodic (cylindrical) boundary conditions and assume that N is an odd integer $N = 2n + 1$. Following [2] we parametrize the Boltzmann weights a, b, c, d of the model as[1]

$$\begin{aligned} a &= \rho\, \vartheta_4(2\eta|\mathsf{q}^2)\, \vartheta_4(u-\eta|\mathsf{q}^2)\, \vartheta_1(u+\eta|\mathsf{q}^2),\\ b &= \rho\, \vartheta_4(2\eta|\mathsf{q}^2)\, \vartheta_1(u-\eta|\mathsf{q}^2)\, \vartheta_4(u+\eta|\mathsf{q}^2),\\ c &= \rho\, \vartheta_1(2\eta|\mathsf{q}^2)\, \vartheta_4(u-\eta|\mathsf{q}^2)\, \vartheta_4(u+\eta|\mathsf{q}^2),\\ d &= \rho\, \vartheta_1(2\eta|\mathsf{q}^2)\, \vartheta_1(u-\eta|\mathsf{q}^2)\, \vartheta_1(u+\eta|\mathsf{q}^2), \end{aligned} \tag{20}$$

and fix the normalization factor

$$\rho = 2\, \vartheta_2(0|\mathsf{q})^{-1}\, \vartheta_4(0|\mathsf{q}^2)^{-1}. \tag{21}$$

With this parametrization, the constraint (5) is equivalent to the condition

$$\eta = \pi/3, \tag{22}$$

which will always be assumed throughout this paper. This still leaves two arbitrary parameters: the (spectral) parameter u and the elliptic nome $\mathsf{q} = \mathrm{e}^{\mathrm{i}\pi\tau}$, $\mathrm{Im}\,\tau > 0$. The variables ζ, γ and x, defined in (6), (7) and (11), can be written as

$$\zeta = \left[\frac{\vartheta_1\left(\frac{2\pi}{3}\middle|\mathsf{q}^2\right)}{\vartheta_4\left(\frac{2\pi}{3}\middle|\mathsf{q}^2\right)}\right]^2, \qquad \gamma = -\left[\frac{\vartheta_1\left(\frac{\pi}{3}\middle|\mathsf{q}^{1/2}\right)}{\vartheta_2\left(\frac{\pi}{3}\middle|\mathsf{q}^{1/2}\right)}\right]^2, \qquad x = \gamma\left[\frac{\vartheta_3\left(\frac{u}{2}\middle|\mathsf{q}^{1/2}\right)}{\vartheta_4\left(\frac{u}{2}\middle|\mathsf{q}^{1/2}\right)}\right]^2, \quad z = \gamma^{-2}. \tag{23}$$

Note that the last expression for x determines our choice of a particular root of the quadratic equation (11).

2.2. The TQ-equation

Any eigenvalue, $T(u)$, of the row-to-row transfer matrix[2] of the 8V-model satisfies Baxter's famous TQ-equation [2]:

$$T(u)Q(u) = \phi(u-\eta)Q(u+2\eta) + \phi(u+\eta)Q(u-2\eta), \tag{24}$$

where, with an account of (21),

$$\phi(u) = \vartheta_1^N(u|\mathsf{q}). \tag{25}$$

[1] We use the notation of [11] for theta-functions $\vartheta_k(u\,|\,\mathsf{q})$, $k = 1, 2, 3, 4$, of the periods π and $\pi\tau$, $\mathsf{q} = \mathrm{e}^{\mathrm{i}\pi\tau}$, $\mathrm{Im}\,\tau > 0$. The theta-functions $\mathsf{H}(v)$, $\Theta(v)$ of the nome q_B used in [2] are given by

$$\mathsf{q}_B = \mathsf{q}^2, \quad \mathsf{H}(v) = \vartheta_1\left(\frac{\pi v}{2\mathsf{K}_B}\middle|\mathsf{q}^2\right), \quad \Theta(v) = \vartheta_4\left(\frac{\pi v}{2\mathsf{K}_B}\middle|\mathsf{q}^2\right),$$

where $\mathsf{K}_B(k)$ is the complete elliptic integral of the first kind with the elliptic modulus $k = \vartheta_2^2(0|\mathsf{q}_B)/\vartheta_3^2(0|\mathsf{q}_B)$.

[2] We use exactly the same definition of the transfer matrix as in [2].

With the parametrization (20), (21) the eigenvalue (9) takes the form

$$T(u) = (a+b)^N = \phi(u), \qquad \eta = \pi/3, \qquad N = 2n+1. \tag{26}$$

Equation (24) for this eigenvalue has been studied in [5]. It has two different solutions [12, 13], denoted by $Q_\pm(u) \equiv Q_\pm(u, \mathsf{q}, n)$, which are *entire* functions of the variable u and obey the following periodicity conditions [2, 14]:

$$\begin{aligned} &Q_\pm(u+\pi) = \pm(-1)^n Q_\pm(u), \qquad Q_\pm(u+\pi\tau) = \mathsf{q}^{-N/2}\,\mathrm{e}^{-\mathrm{i}Nu}\,Q_\mp(u), \\ &Q_\pm(-u) = Q_\pm(u). \end{aligned} \tag{27}$$

The above requirements uniquely determine $Q_\pm(u)$ to within u-independent normalization factors. The solutions $Q_\pm(u)$ satisfy the quantum Wronskian relation [12, 15]

$$Q_+(u+\eta)Q_-(u-\eta) - Q_+(u-\eta)Q_-(u+\eta) = 2i\phi(u)W(\mathsf{q}, n), \tag{28}$$

where $W(\mathsf{q}, n)$ is a function of q and n only (the fact that $W(\mathsf{q}, n)$ does not depend on the variable u follows from (24) and (27)). Note that, taking into account the periodicity (27), one can bring equation (24) to the form

$$\phi(u)Q(u) + \phi(u+2\pi/3)Q(u+2\pi/3) + \phi(u+4\pi/3)Q(u+4\pi/3) = 0. \tag{29}$$

Below it will be more convenient to use the combinations

$$Q_1(u) = (Q_+(u) + Q_-(u))/2, \qquad Q_2(u) = (Q_+(u) - Q_-(u))/2, \tag{30}$$

which are simply related by the periodicity relation

$$Q^{(n)}_{1,2}(u+\pi) = (-1)^n Q^{(n)}_{2,1}(u). \tag{31}$$

Bearing this in mind we will only quote results for $Q_1(u)$, writing it as $Q_1^{(n)}(u)$ to indicate the n-dependence. Introduce new functions $\mathcal{P}_n(u)$ instead of $Q_1^{(n)}(u)$:

$$Q_1^{(n)}(u) = \mathcal{N}(\mathsf{q}, n)\,\vartheta_3(u/2|\mathsf{q}^{1/2})\,\vartheta_4^{2n}(u/2|\mathsf{q}^{1/2})\,\mathcal{P}_n(u), \tag{32}$$

where $\mathcal{N}(\mathsf{q}, n)$ is an arbitrary normalization factor. The analytic properties of $\mathcal{P}_n(u)$ are determined by the periodicity relations (27) and the fact that the eigenvalues $Q^{(n)}_{1,2}(u)$ are entire functions of the variable u. A simple analysis shows that $\mathcal{P}_n(u)$ is an even doubly periodic function of u,

$$\mathcal{P}_n(u) = \mathcal{P}_n(u+2\pi) = \mathcal{P}_n(u+\pi\tau), \qquad \mathcal{P}_n(u) = \mathcal{P}_n(-u), \tag{33}$$

with all its poles[3] (of the order $2n$ and lower) located at the point $u = \pi\tau/2$. Every such function is an nth degree polynomial in the variable x, given by (23) (see section 20.51 of [11]). The coefficients in these polynomials will, of course, depend on the elliptic nome q. Let us now change independent variables from u and q to the variables x and $z = \gamma^{-2}$, defined in (23), and (with a slight abuse of notations) write $\mathcal{P}_n(u)$ as

$$\mathcal{P}_n(x, z) = \sum_{k=0}^{n} r_k^{(n)}(z) x^k. \tag{34}$$

The TQ-equation (29) can be re-written in terms of the polynomials $\mathcal{P}_n(x, z)$. For a fixed value of the nome q, the variable x in (23) is a function of u, so we can write it as $x = x(u)$. Introduce two new variables

$$x_\pm = x\left(u \pm \tfrac{\pi}{3}\right) = \gamma^2/x\left(u \mp \tfrac{2\pi}{3}\right). \tag{35}$$

[3] An apparent pole at $u = \pi + \pi\tau/2$ cancels out because $Q_1^{(n)}(u)$ vanishes at this point as a consequence of the periodicity conditions (27).

They satisfy the relations

$$x_+x_- = \frac{(x-1)^2}{(xz-1)^2}, \qquad x_+ + x_- = \frac{2z(x^2z+1) - x(z^2+4z-1)}{z(xz-1)^2}, \tag{36}$$

which can be easily solved for $x_\pm$ in terms of x and z. The resulting expressions involve a square root from a third-order polynomial in x. It is convenient to define

$$f_\pm = \frac{1}{2} \pm \frac{x(z-1)[(2x-3)z+1]}{2z(x_- - x_+)(xz-1)^2}, \qquad \rho_\pm = \frac{x_\pm - 1}{(1-zx_\pm)x}. \tag{37}$$

With all these new notations, the TQ-equation (29) can now be transformed to its algebraic form

$$\mathcal{P}_n(x,z) = \rho_+ f_-^{2n+1} \mathcal{P}_n\big(z^{-1}x_-^{-1}, z\big) + \rho_- f_+^{2n+1} \mathcal{P}_n\big(z^{-1}x_+^{-1}, z\big). \tag{38}$$

Substituting (34) into the last equation, and expanding it near the point $x = 0$, one immediately obtains a simple relation

$$r_n^{(0)}(z) \equiv \mathcal{P}_n(0,z) = 4^{-n}z^{-1}(z+n(3z-1))\mathcal{P}_n(z^{-1},z) - 4^{-n}z^{-2}(z-1)\frac{\partial \mathcal{P}_n(x,z)}{\partial x}\bigg|_{x=z^{-1}}, \tag{39}$$

quoted here for future references.

2.3. Polynomials $\mathcal{P}_n(x,z)$

Let us now substitute polynomials (34) into the TQ-equation (38). Excluding x_+ and x_- with the help of (36), one can readily see that the rhs of (38) is a rational function of x (indeed, it is a symmetric function of x_+ and x_- and, therefore, can be expressed through two elementary combinations (36)). Writing equation (38) as a polynomial in x and equating its coefficients to zero, one obtains an (overdetermined) system of homogeneous linear equations for $n+1$ unknowns $r_0^{(n)}(z), r_1^{(n)}(z), \ldots, r_n^{(n)}(z)$. All elements of the coefficient matrix for this system are rational functions of the variable z with integer coefficients. This means that with a suitable normalization, all $r_k^{(n)}(z)$, $k = 0, 1, \ldots, n$, can be made polynomials in z with integer coefficients (such that there are no polynomial factors common for all $r_k^{(n)}(z)$). Thus, $\mathcal{P}_n(x,z)$ are two-variable polynomials in x and z with integer coefficients. The first few of them are given in (46) and appendix A below.

Originally, we have calculated [5] these polynomials for $n \leqslant 10$ by directly solving equations (29) and (38) by a combination of analytical and numerical techniques. For larger n this did not appear be to practical due to complexity of intermediate expressions. Subsequently, in the same paper [5], we found a more efficient method for the calculation of $\mathcal{P}_n(x,z)$, based on the partial differential equation (44), discussed below. We have observed that all coefficients of $\mathcal{P}_n(x,z)$ are, in fact, *positive* integers for all $n \leqslant 100$ and suggested that these coefficients might have a combinatorial interpretation (which is yet to be found).

Below we summarize all important properties of $\mathcal{P}_n(x,z)$ discovered in our previous works [1, 5].

Conjecture A ([1, 5]).

(a) *The degrees of the polynomials $r_i^{(n)}(z)$, $i = 0, \ldots, n$, appearing as coefficients in the expansion (34), are given by*

$$\deg\big[r_k^{(n)}(z)\big] \leqslant \lfloor n(n-1)/4 + k/2 \rfloor, \tag{40}$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

(b) If the normalization of $\mathcal{P}_n(x,z)$ is fixed by the requirement

$$r_n^{(n)}(0) = 1, \tag{41}$$

then all polynomials $r_k^{(n)}(z)$, $k = 0, 1, \dots, n$, have positive integers coefficients in their expansions in powers of z.

The normalization (41) will be implicitly assumed throughout the rest of the paper. The most important property of the polynomials $\mathcal{P}_n(x,z)$ is that they satisfy a remarkable linear partial differential equation. This equation can be written in different forms, depending on the choice of independent variables and unknown function. First consider the case of the original variables u and q of the 8V-model. Introduce the functions

$$\Phi_\pm(u,\mathsf{q},n) = \frac{\vartheta_1^{2n+1}(u|\mathsf{q})}{\vartheta_1^n(3u|\mathsf{q}^3)} Q_\pm(u,\mathsf{q},n), \tag{42}$$

where $Q_\pm(u,\mathsf{q},n)$ are eigenvalues of the Q-operators, defined in section 2.2. The analytic properties of $\Phi_\pm(u,\mathsf{q},n)$ in the variable u are determined by (27).

Conjecture B ([5]). *The functions $\Phi_\pm(u,\mathsf{q},n)$, defined by (42), satisfy the non-stationary Schrödinger equation*

$$6q\frac{\partial}{\partial q}\Phi(u,q,n) = \left\{-\frac{\partial^2}{\partial u^2} + 9n(n+1)\wp(3u|q^3) + c(q,n)\right\}\Phi(u,q,n). \tag{43}$$

Here the modular parameter τ plays the role of the (imaginary) time and the time-dependent potential is defined through the elliptic Weierstrass $\wp$-function [11] (our function $\wp(v\,|\,\mathrm{e}^{\mathrm{i}\pi\epsilon})$ has the periods π and $\pi\epsilon$). The constant $c(q,n)$ appearing in (43) is totally controlled by the normalization of $Q_\pm(u)$ and can be explicitly determined once this normalization is fixed (see equations (37) and (38) in [5]). Equation (43) is obviously related to the Lamé differential equation and could be naturally called the 'non-stationary Lamé equation'. This equation arises in various contexts [16, 17] which are not immediately related to this paper.

The differential equation (43) can be equivalently rewritten in an algebraic form for the polynomials $\mathcal{P}_n(x,z)$:

$$\left\{A(x,z)\partial_x^2 + B_n(x,z)\partial_x + C_n(x,z) + T(x,z)\partial_z\right\}\mathcal{P}_n(x,z) = 0, \tag{44}$$

where

$$\begin{aligned}
A(x,z) &= 2x(1+x-3xz+x^2z)(x+4z-6xz-3xz^2+4x^2z^2),\\
B_n(x,z) &= 4(1+x-3xz+x^2z)(x+3z-7xz+3x^2z^2)\\
&\quad + 2nx(1-14z+21z^2-8x^3z^3+3x^2z(3z^2+6z-1)-x(1-9z+23z^2+9z^3)),\\
C_n(x,z) &= n[z(9z-5)+x^2z(3z^2+11z-2)+x(9z^3-38z^2+19z-2)\\
&\quad -4x^3z^3+nz(1-9z-x(9z^2-36z+3)+x^2(3z^2-31z+4)+8x^3z^2)],\\
T(x,z) &= -2z(1-z)(1-9z)(1+x-3xz+x^2z).
\end{aligned} \tag{45}$$

It is fairly easy to prove [5] that the differential equation (43), restricted to a class of functions $\Phi(u,\mathsf{q},n)$ with suitable analytic properties in the variable u, implies the functional equation (29). The non-trivial part of the conjecture B is the fact of *existence* of solutions of (43) with these analytic properties. For equation (44), this translates into a question of existence of solutions, which are polynomials in the variable x.

Equation (44) is extremely useful for finding polynomial solutions, even though the coefficients therein look very complicated. The first polynomials $\mathcal{P}_n(x,z)$ read

$$\mathcal{P}_0(x,z)=1,\qquad \mathcal{P}_1(x,z)=x+3,\qquad \mathcal{P}_2(x,z)=x^2(1+z)+5x(1+3z)+10,$$
$$\mathcal{P}_3(x,z)=x^3(1+3z+4z^2)+7x^2(1+5z+18z^2)+7x(3+19z+18z^2)+35+21z, \tag{46}$$

the next one is given in appendix A. The constant term and leading coefficient in these polynomials (with respect to the variable x) are determined by the following.

Conjecture C ([1, 5]). *The coefficients for the lowest and highest powers of x in $\mathcal{P}_n(x,z)$, corresponding to $k=0$ and $k=n$ in (34), read*

$$\bar{s}_n(z)\equiv r_0^{(n)}(z)=\tau_n(z,-1/3),\qquad s_n(z)\equiv r_n^{(n)}(z)=\tau_{n+1}(z,1/6), \tag{47}$$

where the functions $\tau_n(z,\xi)$ (for each fixed value of the their second argument ξ) are determined by the recurrence relation

$$\begin{aligned}2z(z-1)(9z-1)^2[\log\tau_n(z)]''_z&+2(3z-1)^2(9z-1)[\log\tau_n(z)]'_z\\&+8\left[2n-4\xi-\frac{1}{3}\right]^2\frac{\tau_{n+1}(z)\tau_{n-1}(z)}{\tau_n^2(z)}\\&-[12(3n-6\xi-1)(n-2\xi)+(9z-1)(n-1)(5n-12\xi)]=0,\end{aligned} \tag{48}$$

with the initial condition

$$\tau_0(z,\xi)=1,\qquad \tau_1(z,\xi)=-4\xi+5/3. \tag{49}$$

The functions $\tau_n(z,\xi)$ are polynomials in z for all $n=0,1,2,\ldots,\infty$.

As explained in [5], the partial differential equation (44) leads to descending recurrence relations for the coefficients in (34), in the sense that each coefficient $r_k^{(n)}(z)$ with $k<n$ can be recursively calculated in terms of $r_m^{(n)}(z)$, with $m=k+1,\ldots,n$ and, therefore, can be eventually expressed through the coefficient $r_n^{(n)}(z)$ of the leading power of x. Conditions that this procedure truncates (and thus defines a polynomial, but not an infinite series in negative powers of x) completely determine the starting leading coefficient as a function of z. The above conjecture implies that these truncation conditions are equivalent to the recurrence relation (12) which is a particular case of (48) for $\xi=1/6$. Similar reasonings apply to the coefficient $r_0^{(n)}(z)$ in (34) (the constant term with respect to the variable x).

Note that equation (48) exactly coincides [1] with the recurrence relation for the tau-functions associated with special elliptic solutions of the Painlevé VI equation.

2.4. Quantum Wronskian

We conclude this section with a short analysis of the algebraic form

$$\begin{aligned}\frac{x_-^n}{1-x+x_-(1-xz)}\mathcal{P}_n(x_+,z)\mathcal{P}_n(z^{-1}x_-^{-1},z)&+\frac{x_+^n}{1-x+x_+(1-xz)}\mathcal{P}_n(x_-,z)\mathcal{P}_n\big(z^{-1}x_+^{-1},z\big)\\&=\frac{1}{(x-1)}\left(\frac{z(xz-1)^2(x_+-x_-)^2}{x(z-1)^2}\right)^n\mathsf{W}_n(z)\end{aligned} \tag{50}$$

of the quantum Wronskian relation (28). Here $\mathsf{W}_n(z)$ is related to $W(\mathsf{q},n)$ in (28):

$$W(\mathsf{q},n)=(-1)^n i\,[2\vartheta_1(\pi/3|\mathsf{q})]^{2n+1}\mathcal{N}^2(\mathsf{q},n)\mathsf{W}_n(z). \tag{51}$$

Equation (50) is an algebraic identity valid for arbitrary values of x. Expanding this identity around $x=z^{-1}$, one obtains

$$\mathsf{W}_n(z)=-s_n(z)\mathcal{P}_n(z^{-1},z),\qquad n\geqslant 0, \tag{52}$$

where $s_n(z)$ is defined by (12) (it coincides with the Painlevé VI tau-function $s_n(z) = \tau_{n+1}(z, 1/6)$ defined by (48) for $\xi = 1/6$). Similarly, expanding (50) around $x = 0$ and using (52), one obtains

$$\mathcal{P}_n(1, z) = 4^n s_n(z), \qquad n \geqslant 0. \tag{53}$$

Interestingly the quantity $\mathcal{P}_n(z^{-1}, z)$, entering (52), is also determined by the Painlevé VI recurrence relation (48).

Conjecture D. *The value $\mathcal{P}_n(z^{-1}, z)$ is determined by the recurrence relation (48) with $\xi = 2/3$,*

$$\mathcal{P}_n(z^{-1}, z) = (-4/3)^n z^{-n} \tau_{n+2}(z, 2/3), \qquad n \geqslant 0. \tag{54}$$

Combining the above formulas one obtains the following expression for the quantum Wronskian:

$$\mathsf{W}_n(z) = -(-4/3)^n z^{-n}\, \tau_{n+1}(z, 1/6)\tau_{n+2}(z, 2/3), \qquad n \geqslant 0. \tag{55}$$

3. Eigenvector results

To formulate our results for the eigenvectors (13), we need to define an additional set of polynomials $p_n(y)$ and $q_n(y)$, $n \in \mathbb{Z}$. In principle, these polynomials can be defined by yet another recurrence relation of the Painlevé VI type (though more complicated than (48)) which will be presented elsewhere. For our purposes here, it is much simpler to define these new polynomials $p_n(y)$ and $q_n(y)$ as subfactors of already introduced polynomials $s_n(z)$. We will do this by means of conjecture E, given below. Recall that $s_n(z)$, $n \in \mathbb{Z}$, are defined by equation (12) with the initial conditions $s_0(z) = s_1(z) \equiv 1$.

Conjecture E.

(a) The polynomials $s_{2k+1}(y^2)$ factorize over the integers,

$$s_{2k+1}(y^2) = s_{2k+1}(0)\, p_k(y)\, p_k(-y), \qquad p_k(0) = 1, \quad k \in \mathbb{Z}, \tag{56}$$

where $p_k(y)$ are polynomials in y with integer coefficients, $\deg p_k(y) = k(k+1)$*, such that $p_k'(0) > 0$, $k \geqslant 1$, and $p_k'(0) < 0$, $k \leqslant -2$, where $p_k'(y) = \mathrm{d}p_k(y)/\mathrm{d}y$ denotes the derivative in y. Note that $p_{-1}(y) = p_0(y) \equiv 1$.*

(b) The polynomials $p_k(y)$ possess the symmetry

$$p_k(y) = \left(\frac{1+3y}{2}\right)^{k(k+1)} p_k\left(\frac{1-y}{1+3y}\right), \qquad k \in \mathbb{Z}. \tag{57}$$

(c) The polynomials $s_{2k}(y^2)$ factorize over the integers

$$s_{2k}(y^2) = c_k (1+3y)^{k(k+1)} p_{-k-1}\left(\frac{y-1}{1+3y}\right) q_{k-1}(y), \qquad k \in \mathbb{Z}, \tag{58}$$

where $q_k(y)$ are polynomials in y with integer coefficients, $\deg q_k(y) = k(k+1)$*, $q_k(0) = 1$ and*

$$c_k = 2^{-k(k+2)}, \quad k \geqslant 0; \qquad c_k = 2^{-k^2}(2/3)^{2k+1}, \quad k < 0. \tag{59}$$

(d) Polynomials $q_k(y)$ possess the symmetry

$$q_k(y) = \left(\frac{1+3y}{2}\right)^{k(k+1)} q_k\left(\frac{y-1}{1+3y}\right), \qquad k \in \mathbb{Z}. \tag{60}$$

A few first polynomials $s_n(z)$, $p_n(y)$ and $q_n(y)$ are listed in appendix A. Let us mention one simple, but important corollary of the above conjecture. The lhs of (58) is an even function of the variable y. Combining this fact with the symmetry relation (57), one immediately deduces that $q_k(y)$ is also an even function:

$$q_k(-y) = q_k(y), \qquad k \in \mathbb{Z}. \tag{61}$$

In conjecture 1, given in the introduction, we have stated an explicit expression (19) for the norm $|\Psi_-|^2$ of the eigenvector Ψ_- as a function of the parameter ζ entering the Hamiltonian (1). Note that using the factorization and symmetry properties (56)–(61), one can show that the rescaled norm

$$N_n(\zeta) = (\zeta^2 + 3)^{-n(n+1)/2}\, |\Psi_-|^2 \tag{62}$$

is invariant with respect to the full S_3 symmetry group generated by the substitutions (8), which is a well-expected result. Further, is easy to see that modulo a trivial numerical factor, the expression for the norm remains unchanged upon the replacement $n \to -n-1$, which corresponds to a negation of the length of the chain, $N \to -N$. In other words, the norm is an even function of the length of the chain. It would be interesting to understand a reason of this phenomenon.

We are now ready to present further conjectures on the properties of the eigenvectors.

Conjecture 2. *The component of the eigenvector Ψ_- with one spin down is given by*

$$\psi_{0\ldots001} = \frac{1}{N}\zeta^{n(n-1)/2}\bar{s}_n(\zeta^{-2}), \qquad N = 2n+1, \tag{63}$$

where $\bar{s}_n(z)$ is defined by (47).

A few first polynomials $s_n(z)$ and $\bar{s}_n(z)$ are listed in appendix A.

Conjecture 3. *The component of the eigenvector Ψ_- with all spins down is given by*

$$\psi_{11\ldots11} = \zeta^{n(n+1)/2}s_n(\zeta^{-2}), \qquad N = 2n+1, \tag{64}$$

where $s_n(z)$ is defined by (47).

It is interesting to note that, to within a simple power of ζ, the above two components of the eigenvector precisely coincide with the constant term and leading coefficients of the polynomial $\mathcal{P}_n(x,z)$, which is simply connected (32) with the corresponding eigenvalue of the Q-operator. We believe that this fact certainly deserves further studies.

Finally, consider components of Ψ_- with alternating (up and down) spins in the chain

$$A_n(\zeta) = \Psi_{00101\ldots01}, \quad \text{for odd } n; \qquad A_n(\zeta) = \Psi_{0101\ldots011} \quad \text{for even } n. \tag{65}$$

In the case $\zeta = 0$, these are largest components of the eigenvector.

Conjecture 4. *The components of Ψ_- with alternative spins are given by*

$$\begin{aligned} A_{2k}(\zeta) &= 2^{k(2-k)}(3+\zeta)^{k(k-1)}\zeta^{k(k-1)}p_{k-1}\left(\frac{1-\zeta}{3+\zeta}\right)q_{k-1}(\zeta^{-1}) \\ A_{2k+1}(\zeta) &= 2^{-k^2}(3+\zeta)^{k(k+1)}\zeta^{k(k-1)}p_k\left(\frac{1-\zeta}{3+\zeta}\right)q_{k-1}(\zeta^{-1}). \end{aligned} \tag{66}$$

A few polynomials $p_n(y)$, $q_n(y)$ and $A_n(\zeta)$ are listed in appendix A. As noted before, the case $\zeta = 0$ corresponds to the XXZ-model with $\Delta = -1/2$. It is known [7, 18–20] that in this particular case, the values of the components (65), normalized by (18), coincide with the number of alternating sign matrices

$$A_n(0) = A_n = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} \tag{67}$$

calculated in [21]. Using this result in (66) one easily obtains for $n \geqslant 0$,

$$p_n\left(\frac{1}{3}\right) = \left(\frac{2}{3}\right)^{n(n+1)} \prod_{k=0}^{n} \frac{(2k)!(6k+1)!}{(4k)!(4k+1)!}, \tag{68}$$

$$\zeta^{n(n+1)} q_n(\zeta^{-1})|_{\zeta=0} = 2^{-n-1} \prod_{k=0}^{n} \frac{(2k+1)!(6k+4)!}{(4k+2)!(4k+3)!}. \tag{69}$$

Apparently one can derive these expressions directly from the definitions of the polynomials $p_n(y)$ and $q_n(y)$, given in conjecture E; however, we postpone this to a future publication.

Finally, we mention one amusing observation connected with expressions (66). It is not difficult to analytically derive an asymptotic expansion $A_n^{(\mathrm{asymp})}(\zeta)$ which correctly reproduces first terms of the expansion of $A_n(\zeta)$ for small ζ up to the order $O(\zeta^{2n})$,

$$A_n(\zeta) = A_n^{(\mathrm{asymp})}(\zeta) + O(\zeta^{2n}), \qquad \zeta \to 0. \tag{70}$$

Analytically continuing this asymptotic expansion to $n = 0$,

$$A_n^{(\mathrm{asymp})}(\zeta)|_{n=0} = 1 - \zeta^2 - 3\zeta^4 - 15\zeta^6 - 86\zeta^8 - 534\zeta^{10} - 3478\zeta^{12} - \cdots, \tag{71}$$

and plugging its coefficients into Sloane's integer sequences database (in a search for a discovery), we found that they only 'slightly' mismatched numbers of lattice animals made of n three-dimensional cubes [22], which are 1, 3, 15, 86, 534, 3481, ... Of course, it would be extremely weird if they matched.

4. Conclusion and outlook

In this paper we have demonstrated that a particular anisotropic XYZ-model, defined by (1) and (2), is deeply related to the theory of the Painlevé VI equation. We have proposed exact expressions for the norm (conjecture 1) and certain components of the ground state eigenvectors (conjectures 2, 3, 4). The results are expressed in terms of the tau-functions associated with the special elliptic solutions of the Painlevé VI equation [1].

In this connection, it is useful to mention other celebrated appearances of Painlevé transcendents in mathematical physics. The most prominent examples include the two-dimensional Ising model [23], the problem of isomonodromic deformations of the second-order differential equations [24] and the field theory approach to dilute self-avoiding polymers on a cylinder [25–29]. The latter problem is connected with the massive sine-Gordon model at the supersymmetric point (where the ground state energy vanishes exactly due to supersymmetry). Our previous work [5] grew from attempts to develop an alternative approach to this polymer problem based on the lattice theory. It turns out that all non-trivial information about dilute polymer loops is contained in the ground state eigenvalues [5] of the Q-operator for the 8V-model on a periodic chain of an odd length, connected with the special XYZ-model, considered in this paper. In [5] we have found that these eigenvalues can be uniquely determined as certain polynomial solutions $\mathcal{P}_n(x, z)$ of the partial differential equation (44) (recall that the variables x and z are connected to the original spectral parameter u and the elliptic nome q, respectively; see (23)). So far we have not ultimately understood the role of this equation in the Painlevé VI theory, but there is no doubt that there are profound connections. For example, one-variable specialization of $\mathcal{P}_n(x, z)$ at particular values of x (which remain polynomials in the variable z) are connected with the tau-functions associated with the Picard solutions of the Painlevé VI equation (see equations (53) and (54)). The same property is also enjoyed by the coefficients in the expansion of $\mathcal{P}_n(x, z)$ in powers of x. Note that namely these coefficients provide 'construction materials' in the expression for the ground state eigenvectors (see equations (19), (56)–(66)). Most of our results are conjectures and it is, of course, desirable to obtain their proofs. Another outstanding problem is an algebraic construction of the Q-matrix. As noted in [30], the method used in [2] for the construction of the Q-matrix cannot be executed in its full strength for $\eta = \pi/3$ and odd values of N, since some axillary $\mathbf{Q}$-matrix, $\mathbf{Q}_R(u)$, in [2] is not invertible in the full 2^N-dimensional space of states of the model. Apparently, the construction of [2] could be modified to resolve this difficulty. We hope to address this question in the future.

It is reasonable to expect that mathematical structures, similar to those described above (namely, the partial differential equations and Painlevé-type recurrence relations), should manifest itself in other problems, closely related to the 8V-model with $\eta = \pi/3$. The most immediate candidate is the corresponding 'eight-vertex solid-on-solid' (8VSOS) model which belongs to a rich variety of algebraic constructions associated with the 8V-model [8]. Recently, Rosengren [9], motivated by considerations of the 3-coloring problem [10], studied precisely this 8VSOS-model with $\eta = \pi/3$ in the case of the domain wall boundary conditions. In equation (8.11) of his paper [9], he introduced a set of two-variable polynomials (defined recursively), related to the partition functions of the 8VSOS model on finite lattices. A detailed inspection of these polynomials suggests that they satisfy a partial differential equation, which is completely analogous (though not identical) to our equation (44)! This new differential equation is presented in appendix B.

Furthermore, we found that some one-variable specialization of Rosengren's polynomials, also defined in [9], satisfy a recurrence relation, which is extremely similar to the Painlevé VI type recurrence relation (48) of this paper. This new relation is also presented in appendix B. As yet it is not written in a canonical form for Painlevé VI type relations; however, we expect that it could be brought to such form by a suitable change of variables. We also expect that this new recurrence relation for the 8VSOS-model can be connected with the Picard elliptic solutions of the Painlevé VI following the method of our previous paper [1]. It seems it would be extremely interesting to further compare our results with those of [9].

Acknowledgments

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Appendix A. Polynomials $\mathcal{P}_n(x, z)$, $s_n(z)$, $\bar{s}_n(z)$, $p_n(z)$ and $q_n(z)$

In this appendix, we present explicit expressions for the polynomials $\mathcal{P}_n(x, z)$, $s_n(z)$, $\bar{s}_n(z)$, $p_n(z)$ and $q_n(z)$ for small values of the their index n.

The two-variable polynomials $\mathcal{P}_n(x, z)$, represented by equation (34), are defined as solutions of the TQ-equation (38) normalized by (41). These polynomials can be efficiently calculated from the differential equation (44):

$$\begin{aligned}
&\mathcal{P}_0(x,z)=1, \qquad \mathcal{P}_1(x,z)=x+3, \qquad \mathcal{P}_2(x,z)=x^2(1+z)+5x(1+3z)+10,\\
&\mathcal{P}_3(x,z)=x^3(1+3z+4z^2)+7x^2(1+5z+18z^2)+7x(3+19z+18z^2)+35+21z,\\
&\mathcal{P}_4(x,z)=x^4(1+6z+18z^2+30z^3+9z^4)+9x^3(1+8z+38z^2+152z^3+57z^4)\\
&\qquad +18x^2(2+19z+111z^2+217z^3+99z^4)+12x(7+72z+171z^2+198z^3)\\
&\qquad +18(7+14z+11z^2).
\end{aligned} \tag{A.1}$$

The polynomials $s_n(z)$ and $\overline{s}_n(z)$, where $n \in \mathbb{Z}$, are defined by equation (47) and the recurrence relations (48), (49). For non-negative n they coincide with the coefficients of the highest and lowest powers of x in $\mathcal{P}_n(x,z)$, corresponding to $k=n$ and $k=0$ in the expansion (34):

$$\begin{aligned}
&s_{-5}=\tfrac{1}{256}(81+1215z+10\,206z^2+64\,638z^3+353\,565z^4+544\,563z^5+352\,836z^6),\\
&s_{-4}=\tfrac{1}{64}(27+270z+1620z^2+7938z^3+3969z^4),\\
&s_{-3}(z)=\tfrac{1}{16}(9+54z+225z^2),\\
&s_{-2}(z)=\tfrac{1}{4}(3+9z),\\
&s_{-1}(z)=s_0(z)=s_1(z)=1,\\
&s_2(z)=1+z,\\
&s_3(z)=1+3z+4z^2,\\
&s_4(z)=1+6z+18z^2+30z^3+9z^4,\\
&s_5(z)=1+10z+51z^2+168z^3+355z^4+318z^5+121z^6
\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}
&\overline{s}_0(z)=1, \qquad \overline{s}_1(z)=3, \qquad \overline{s}_2(z)=10,\\
&\overline{s}_3(z)=35+21z,\\
&\overline{s}_4(z)=126+252z+198z^2,\\
&\overline{s}_5(z)=462+1980z+3960z^2+4004z^3+858z^4,\\
&\overline{s}_6(z)=1716+12\,870z+47\,190z^2+105\,820z^3+143\,520z^4+90\,558z^5+24\,310z^6.
\end{aligned} \tag{A.3}$$

The polynomials $p_n(y)$ and $q_n(y)$ are defined by the factorization relations (56) and (58) in conjecture E:

$$\begin{aligned}
&p_{-3}(y)=1-3y+12y^2-30y^3+81y^4-63y^5+66y^6,\\
&p_{-2}(y)=1-2y+5y^2,\\
&p_{-1}(y)=p_0(y)=1,\\
&p_1(y)=1+y+2y^2,\\
&p_2(y)=1+2y+7y^2+10y^3+21y^4+12y^5+11y^6,\\
&p_3(y)=1+3y+15y^2+35y^3+105y^4+195y^5+435y^6\\
&\qquad +555y^7+840y^8+710y^9+738y^{10}+294y^{11}+170y^{12}
\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}
&q_{-3}(y)=1+3y^2+39y^4+21y^6,\\
&q_{-2}(y)=1+3y^2,\\
&q_{-1}(y)=q_0(y)=1,\\
&q_1(y)=1+3y^2,\\
&q_2(y)=1+8y^2+29y^4+26y^6,\\
&q_3(y)=1+15y^2+112y^4+518y^6+1257y^8+1547y^{10}+646y^{12}.
\end{aligned} \tag{A.5}$$

Finally, we list the polynomials $A_n(\zeta)$ from expressions (66) for the alternative spin components (65),

$$
\begin{aligned}
&A_1(\zeta) = 1, \qquad A_2(\zeta) = 2, \qquad A_3(\zeta) = 7+\zeta^2, \qquad A_4(\zeta) = 2(3+\zeta^2)(7+\zeta^2),\\
&A_5(\zeta) = (3+\zeta^2)(143+99\zeta^2+13\zeta^4+\zeta^6),\\
&A_6(\zeta) = 2(26+29\zeta^2+8\zeta^4+\zeta^6)(143+99\zeta^2+13\zeta^4+\zeta^6),\\
&A_7(\zeta) = (26+29\zeta^2+8\zeta^4+\zeta^6)\\
&\qquad\times(8398+14\,433\zeta^2+7665\zeta^4+2010\zeta^6+240\zeta^8+21\zeta^{10}+\zeta^{12}),\\
&A_8(\zeta) = 2(646+1547\zeta^2+1257\zeta^4+518\zeta^6+112\zeta^8+15\zeta^{10}+\zeta^{12})\\
&\qquad\times(8398+14\,433\zeta^2+7665\zeta^4+2010\zeta^6+240\zeta^8+21\zeta^{10}+\zeta^{12})\\
&A_9(\zeta) = 2(646+1547\zeta^2+1257\zeta^4+518\zeta^6+112\zeta^8+15\zeta^{10}+\zeta^{12})\\
&\qquad\times(1411\,510+4598\,551\zeta^2+5518\,417\zeta^4+3530\,124\zeta^6+1331\,064\zeta^8\\
&\qquad+327\,810\zeta^{10}+53\,382\zeta^{12}+5820\zeta^{14}+506\zeta^{16}+31\zeta^{18}+\zeta^{20}).
\end{aligned}
\tag{A.6}
$$

Appendix B. Comments on the 8VSOS-model

Recently, Rosengren [9], motivated by considerations of the 3-coloring problem [10], studied the 8VSOS-model with $\eta = \pi/3$ in the case of the domain wall boundary conditions. In equation (8.11) of his paper [9], he introduced a set of two-variable polynomials, related to the partition functions of the 8VSOS model on finite lattices. Here we denote these polynomials as $P_n^{(\mathrm{SOS})}(t,s)$, adding the superscript 'SOS' to indicate their relevance to the 8VSOS-model[4]. A detailed inspection of these polynomials suggests that

Conjecture 5. *The polynomials $P_n^{(\mathrm{SOS})}(t,s)$, for even values of $n = 0, 2, 4\ldots$, are uniquely determined (up to a numerical normalization) by the the following partial differential equation in the variables x and s:*

$$\left\{A^{(\mathrm{SOS})}(t,s)\partial_t^2 + B_n^{(\mathrm{SOS})}(t,s)\partial_t + C_n^{(\mathrm{SOS})}(t,s) + T^{(\mathrm{SOS})}(t,s)\partial_s\right\} P_n^{(\mathrm{SOS})}(t,s) = 0, \tag{B.1}$$

where

$$
\begin{aligned}
A^{(\mathrm{SOS})}(t,s) &= 2t(1-t)(1+2s-t)(s+2s^2-2t-st)(2t+st-s),\\
B_n^{(\mathrm{SOS})}(t,s) &= -4(2+s)^2t^4 - 4(2+s)(-2s^2+s^2n+3sn-5s-3+n)t^3\\
&\quad + (40sn+8s^4n+40s^3n+60s^2n+8n-8s^4-60s^2-8-44s^3-36s)t^2\\
&\quad - 4(1+2s)s(-s^2+3s^2n+5sn-3s+3n-2)t + 4ns^2(1+2s)^2,\\
C_n^{(\mathrm{SOS})}(t,s) &= 2n(2+s)^2(1+n)t^3 - 4n(2+s)(1+s)^2(1+n)t^2\\
&\quad + ns(8s^3+4s^3n+26s^2+15s^2n+18s+18sn+2+5n)t\\
&\quad - 4n(1+2s)s(s+sn+1),\\
T^{(\mathrm{SOS})}(t,s) &= 4(1-s^2)s(2+s)(1+2s)t.
\end{aligned}
\tag{B.2}
$$

A similar differential equation exists for odd values of n. Obviously, the above property is a counterpart of the partial differential equation (44) in the main text of this paper.

[4] Here we use the variables t and s instead of x and ζ used in [9]. These variables are similar, *but not identical*, to our variables x and z in (23). In particular, the variable t (which corresponds to x in [9]) is also connected with the spectral parameter, while the variable s (denoted as ζ in [9]) is related to the elliptic nome q.

Next, define one-variable polynomials[5]

$$p_n^{(\mathrm{SOS})}(s) = (1+2s)^{[\frac{n^2}{4}]-[\frac{n}{2}]}\left(1+\frac{s}{2}\right)^{[\frac{(n-1)^2}{4}]} P_n^{(\mathrm{SOS})}(1+2s, s), \tag{B.3}$$

where $[x]$ denotes the integer part of x. We suggest that

Conjecture 6. *The polynomials $p_n^{(\mathrm{SOS})}(s)$ satisfy the following recurrence relations:*

$$\begin{aligned} &s(s-1)^2(s+2)(2s+1)\partial_s^2 \log p_n^{(\mathrm{SOS})}(s) + 2(s-1)(s^3-3s^2-6s-1)\partial_s \log p_n^{(\mathrm{SOS})}(s) \\ &\quad - 4(2n+1)(2n+3)\frac{p_{n+1}^{(\mathrm{SOS})}(s)p_{n-1}^{(\mathrm{SOS})}(s)}{(p_n^{(\mathrm{SOS})}(s))^2} \\ &\quad + (22n^2+35n+18)s^2 + (46n^2+98n+42)s + 13n^2+29n+12 = 0, \end{aligned} \tag{B.4}$$

with the initial condition $p_0(s) = 1$, $p_1(s) = 1+3s$.

The structure of relation (B.4) is very similar to that of the recurrence relation (48) for the tau-functions of the Painlevé VI equation. It should be noted that equation (B.4) is not written in a canonical form for such recurrence relations. Nevertheless, we expect that it could be brought to such form by a suitable change of variables. We also expect that (B.4) can be connected with the Picard elliptic solutions of the Painlevé VI following the method of our previous paper [1].

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[5] These polynomials are simply related to those introduced in [9]:

$$p_n^{(\mathrm{SOS})}(s) = (1+2s)^{[\frac{n^2}{4}]}\left(1+\frac{s}{2}\right)^{[\frac{(n-1)^2}{4}]} p_n^{(R)}(s),$$

where $p_n^{(R)}(s)$ are defined by the first unnumbered equation after the proposition 3.12 in [9]. We thank H Rosengren for sending us the modified definition (B.3).

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